Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.7. Linear programming

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Linear Programming

Product	P_1	P_2	Global Resurs
Resurs S_1	2	2	14
Resurs S_2	1	2	8
Resurs S_3	4	0	16
Profit/unit	2	3	

Decision variable:

x_1, x_2 - value of production product P_1, P_2 respectively

Goal function:

$$F(x_1, x_2) = 2x_1 + 3x_2$$

Constrains:

$$2x_1 + 2x_2 \leq 14$$
$$x_1 + 2x_2 \leq 8$$
$$4x_1 + 0x_2 \leq 16$$
$$x_1 \geq 0, x_2 \geq 0$$

Linear Programming

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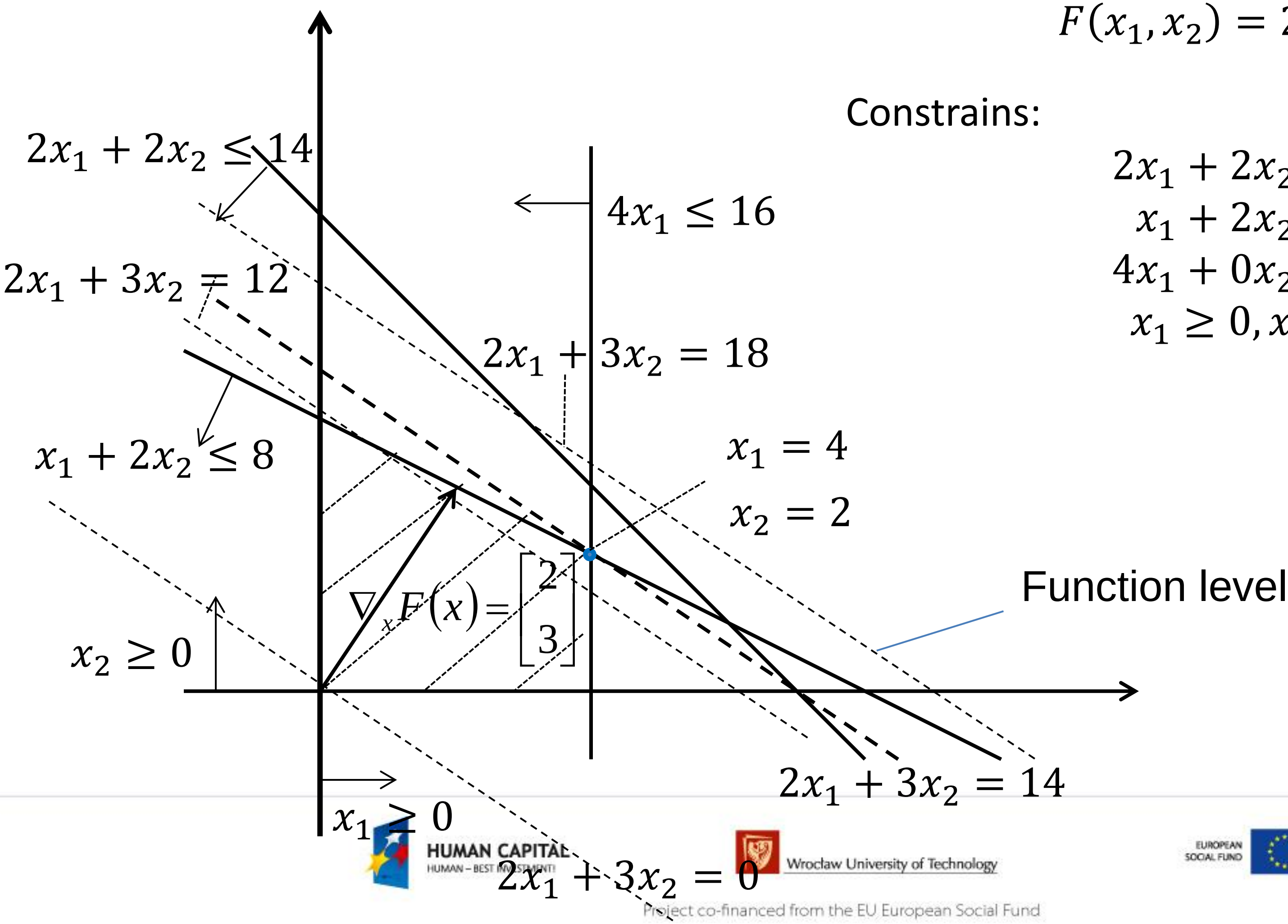
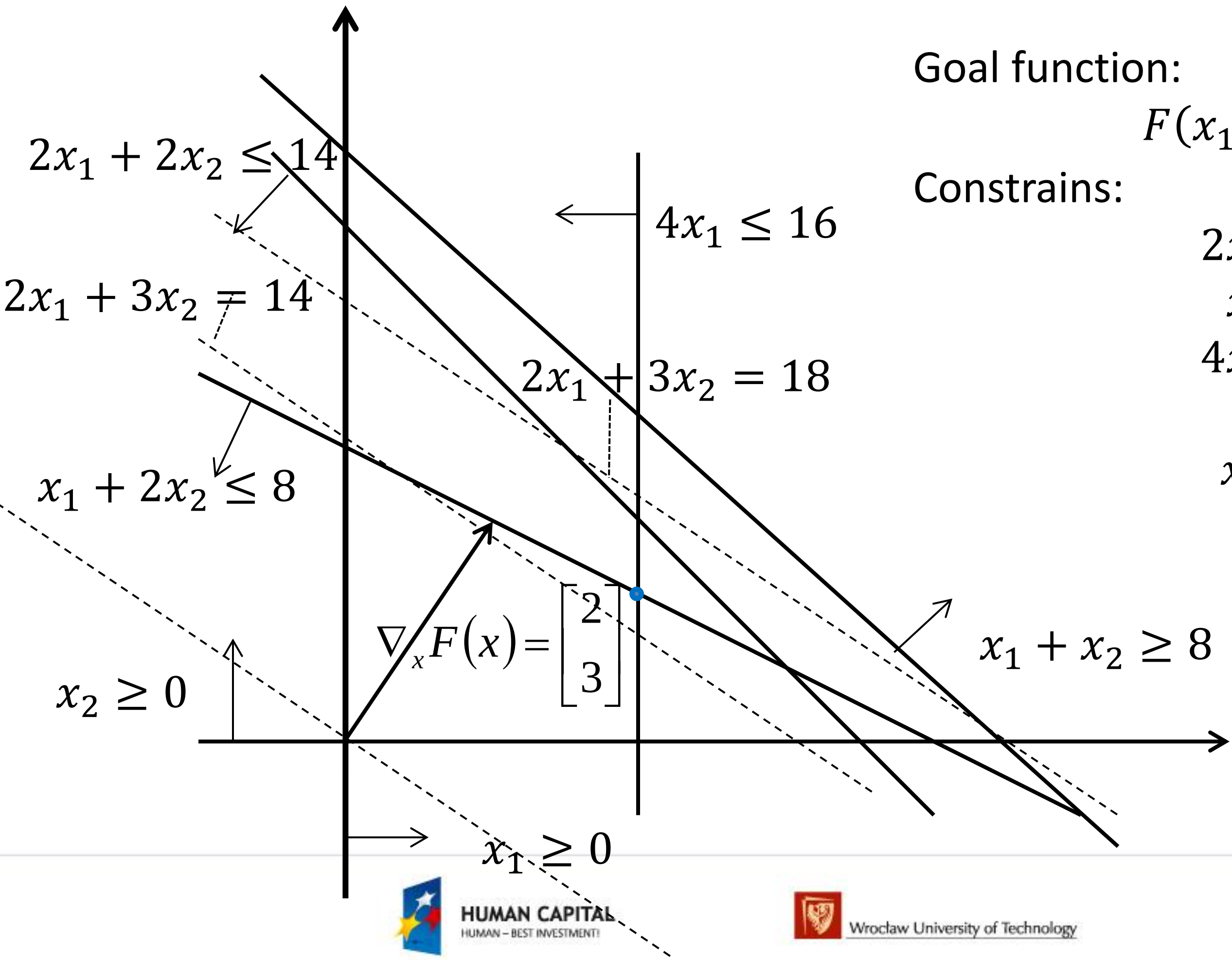


Linear Programming

Problem like before but sum of product greater then 8,



Goal function:

$$F(x_1, x_2) = 2x_1 + 3x_2$$

Constrains:

$$2x_1 + 2x_2 \leq 14$$
$$x_1 + 2x_2 \leq 8$$
$$4x_1 + 0x_2 \leq 16$$
$$x_1 + x_2 \geq 8 \quad !!!!!!$$
$$x_1 \geq 0, x_2 \geq 0$$

Solution set is empty set

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Linear Programming

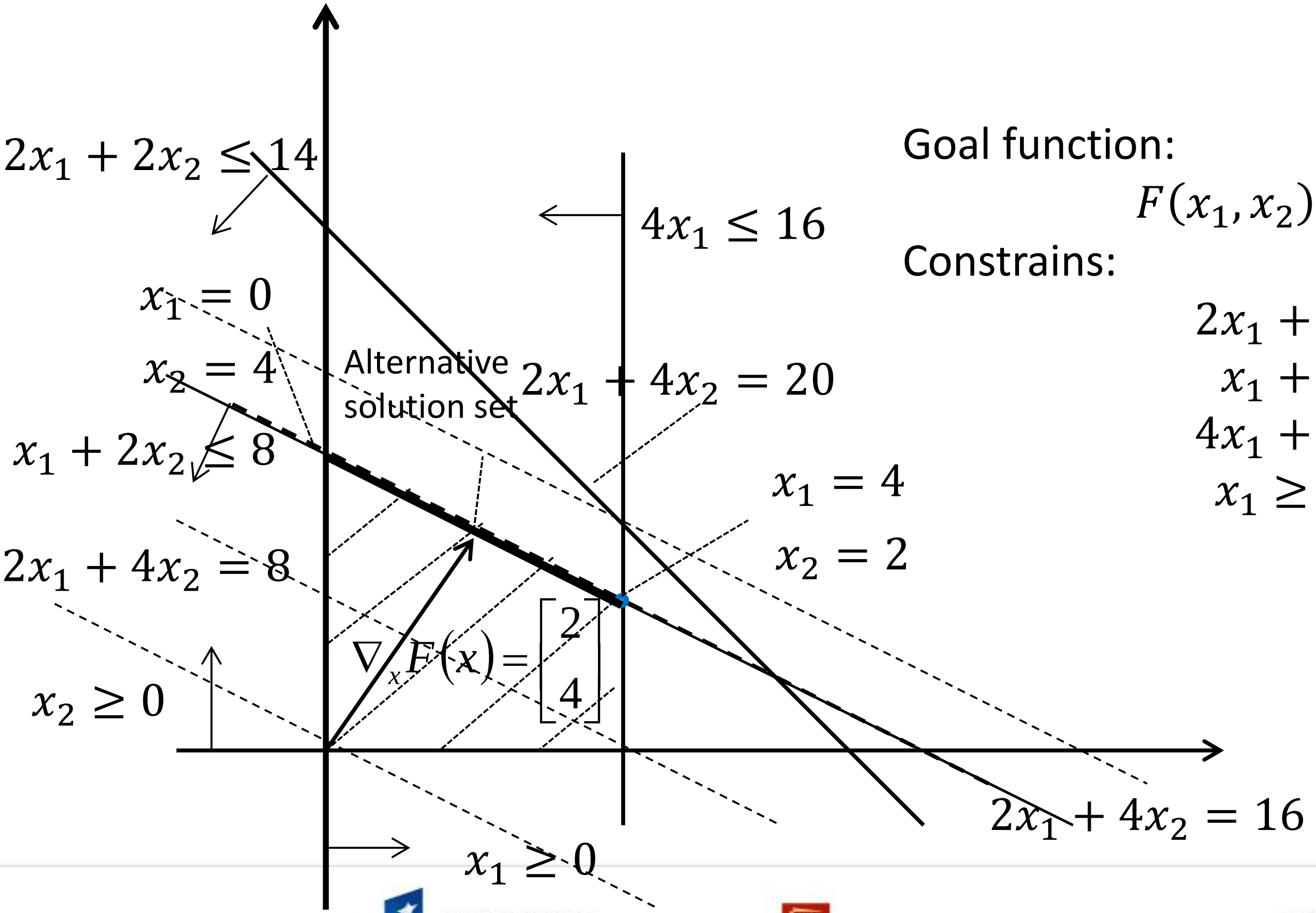
Problem like at the begining, but profit of product P_2 is 4

Goal function:

$$F(x_1, x_2) = 2x_1 + 4x_2$$

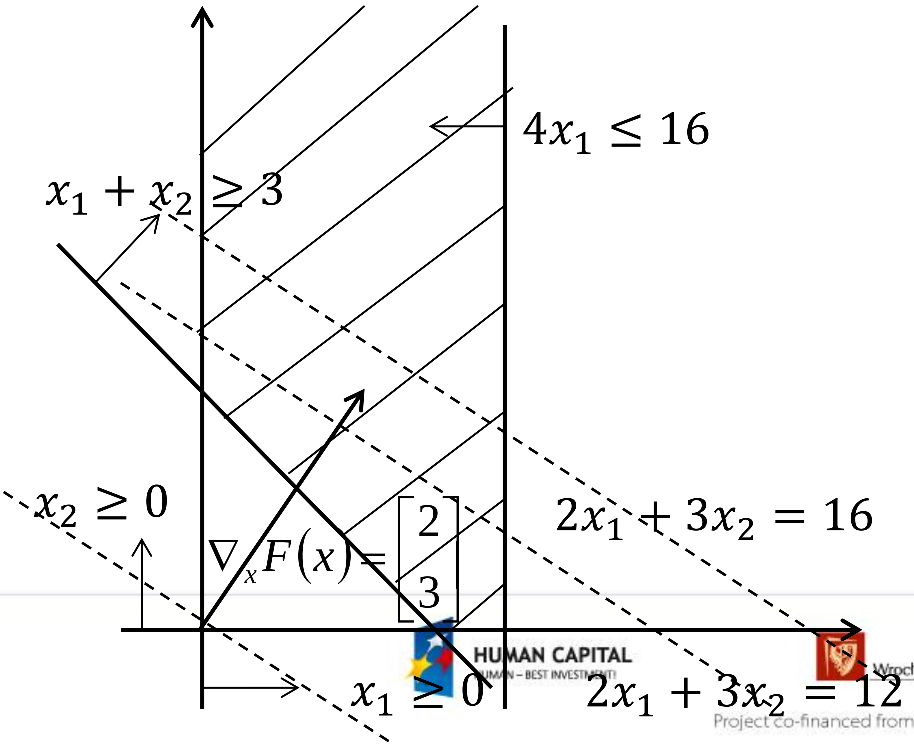
Constrains:

$$2x_1 + 2x_2 \leq 14$$
$$x_1 + 2x_2 \leq 8$$
$$4x_1 + 0x_2 \leq 16$$
$$x_1 \geq 0, x_2 \geq 0$$



Product	P_1	P_2	Global Resurs
Resurs S_1	2	2	Not limited
Resurs S_2	1	2	Not limited
Resurs S_3	4	0	16
Profit/unit	2	3	

Sum of products greater than 3.



Goal function:

$$F(x_1, x_2) = 2x_1 + 3x_2$$

Constrains:

$$4x_1 + 0x_2 \leq 16$$

$$x_1 + x_2 \geq 3$$

$$x_1 \geq 0, x_2 \geq 0$$

Non constrained solution

General problem formulation

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$\mathcal{D}_x = \{x \in R^S, \varphi_l(x) = 0, l = 1, 2, \dots, L, \psi_m(x) \le 0, m = 1, 2, \dots, M\}$$

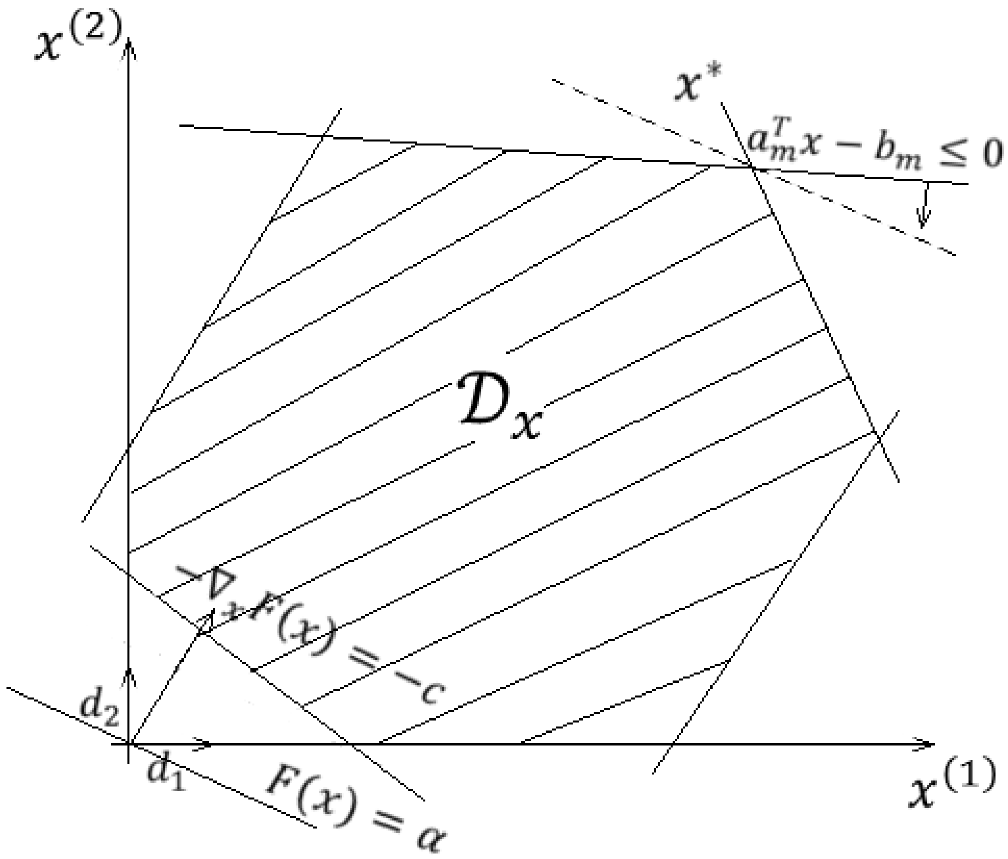
$$F(x) = c^T x = \sum_{s=1}^{S} c_s x^{(s)}$$

$$\varphi_l(x) = a_l^T - b_l = \sum_{s=1}^{S} a_{ls} x^{(s)} - b_l = 0 \quad l = 1, 2, \dots, L$$

$$\psi_m(x) = a_m^T x - b_m \le 0 = \sum_{s=1}^{S} a_{ms} x^{(s)} - b_m \le 0 \quad m = 1, 2, \dots, M$$

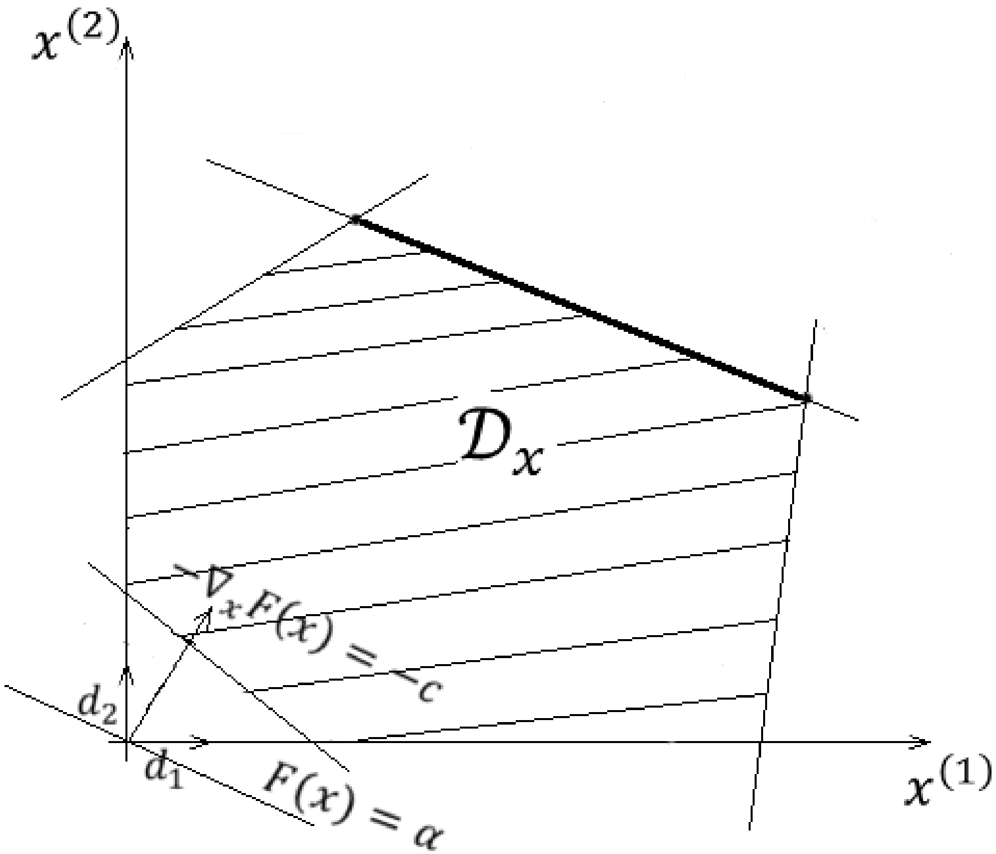
$$x^{(s)} \ge 0 \quad s = 1, 2, \dots, S$$

Geometric view



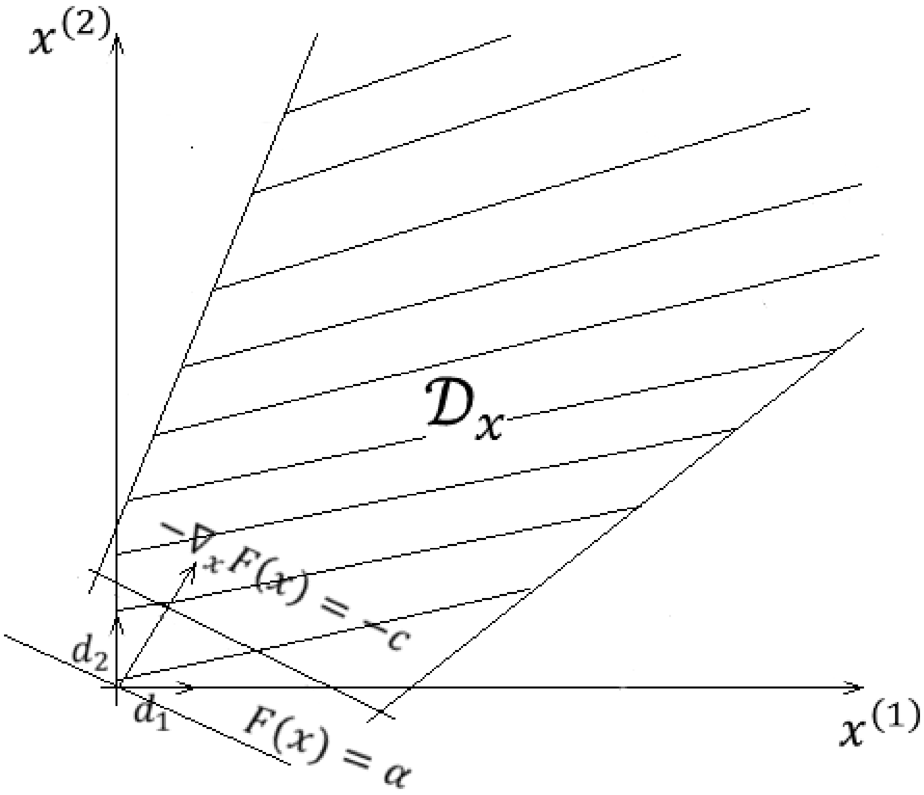
1. Solution is located on a vertex

Geometric view



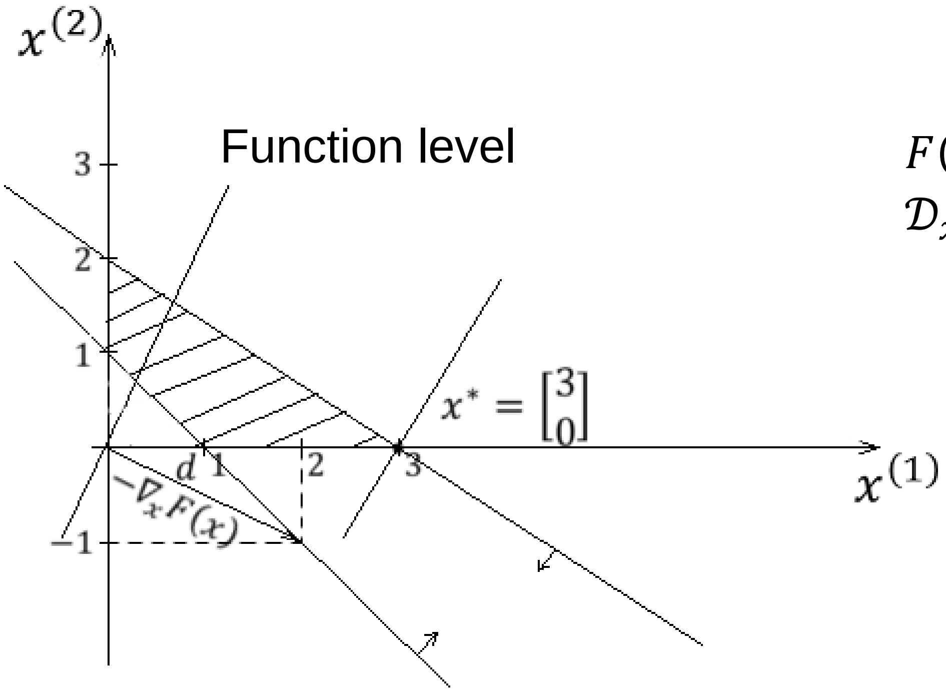
2. Solution is located on an edge

Geometric view



3. Unbounded solution

Example



$$F(x) = -2x^{(1)} + x^{(2)}$$

$$\mathcal{D}_x = \{x \in R^2, -x^{(1)} - x^{(2)} + 1 \leq 0,$$
$$2x^{(1)} + 3x^{(2)} - 6 \leq 0,$$
$$x^{(1)}, x^{(2)} \geq 0\}$$

$$\nabla_x F(x) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad -\nabla_x F(x) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Standard form

$$F(x) = c^T x$$

$$\mathcal{D}_x = \{x \in R^S, a_l^T x - b_l = 0, l = 1,2,\dots,L, a_m^T x - b_m \le 0, m = 1,2,\dots,M, x^{(s)} \ge 0 \quad s = 1,2,\dots,S\}$$

Standard form

A: $\mathcal{D}_X = \{x \in R^S, Ax - b = 0_L, x \ge 0_S\}$

or

B: $\mathcal{D}_x = \{x \in R^S, Ax - b \le 0_L, x \ge 0_S\}$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_S \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_L \end{bmatrix}, \quad x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(S)} \end{bmatrix}, \quad A_{SxL} = \begin{bmatrix} a_{11} & \cdots & a_{1S} \\ \vdots & \ddots & \vdots \\ a_{L1} & \cdots & a_{LS} \end{bmatrix}$$

$B \rightarrow A$

1° $a_l^T x - b_l \leq 0$ we introduce slack variables $x_{S+1} \geq 0$

$$a_l^T x + x_{S+1} - b_l = 0$$

or

$$\bar{a}_l = \begin{bmatrix} a_{L1} \\ \vdots \\ a_{LS} \\ 1 \end{bmatrix}, \bar{x} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(S)} \\ x^{(S+1)} \end{bmatrix}, \bar{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_S \\ 0 \end{bmatrix}$$

$$F(x) = \bar{c}^T \bar{x}$$

$$\bar{a}_l{}^T \bar{x} - b_l = 0$$

$A \to B$

2° $a_l^T x - b_l = 0 \equiv \begin{cases} a_l^T x - b_l \le 0 \\ -a_l^T x + b_l \le 0 \end{cases}$

3° $x^{(s)}$ – is unbounded

$x^{(s)} = x^{(s)\prime} - x^{(s)\prime\prime}$

$x^{(s)\prime} \ge 0, \qquad x^{(s)\prime\prime} \ge 0$

$$\bar{x} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{s-1} \\ x^{(s)\prime} \\ x^{(s)\prime\prime} \\ x^{s+1} \\ \vdots \\ x^{s} \end{bmatrix}, \qquad \bar{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{s-1} \\ c_s \\ -c_s \\ c_{s+1} \\ \vdots \\ c_s \end{bmatrix}, \qquad \bar{a}_l = \begin{bmatrix} a_{l1} \\ \vdots \\ a_{ls-1} \\ a_{ls} \\ -a_{ls} \\ a_{ls+1} \\ \vdots \\ a_{ls} \end{bmatrix}$$

$$F(x) = -2x^{(1)} + x^{(2)} \quad \rightarrow \quad F(x) = -2x^{(1)} + x^{(2)} + 0x^{(3)} + 0x^{(4)}$$

$$-x^{(1)} - x^{(2)} + 1 \leq 0 \quad \rightarrow \quad -x^{(1)} - x^{(2)} + x^{(3)} + 1 = 0 \qquad x^{(3)} \geq 0$$

$$2x^{(1)} + 3x^{(2)} - 6 \leq 0 \quad \rightarrow \quad 2x^{(1)} + 3x^{(2)} + x^{(4)} - 6 = 0 \qquad x^{(4)} \geq 0$$

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \qquad \mathrm{b} = \begin{bmatrix} -1 \\ 6 \end{bmatrix},$$

$$x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ x^{(4)} \end{bmatrix}, \qquad c = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$F(x) = c^T x$$

$$\mathcal{D}_X = \{x \in R^S\colon \; Ax - b = 0_L, \qquad x \geq 0\}$$

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b) - \mu^T x$$

$$\nabla_x L(x, \lambda) = c + A^T \lambda - \mu = 0_S$$

$$\nabla_\lambda L(x, \lambda) = Ax - b = 0_L \quad \text{!!!}$$

$$\mu^T \nabla_\mu L = \mu^T x = 0$$

$$\mu \geq 0_S$$

$$x \geq 0_S$$

Feasible solution $x \in \mathcal{D}_x$

$$Ax = b \qquad Rz(A) = L \qquad S \geq L$$

Basic solution

$$x^B = B^{-1}b \qquad B \text{ – matrix containing } L \text{ kolumn of the matrix } A$$

the total number of basic solutions is at most:

$$\frac{S!}{L!\,(S-L)!}$$

Basic feasible solution $x^B \geq 0_L$

Not degenerated basic feasible solution $x^B > 0_L$

The simplex method

1. Generation of initial feasible solution
2. Convergence criteria – the stopping condition
3. Changing the basis
4. Dealing with degenerate basic solutions

$Ax = b \quad x_B = B^{-1}b$– the basic solution

$$x_B = B^{-1}Ax$$

$$Ax - b = 0_L \quad /B^{-1}$$

$$B^{-1}Ax - B^{-1}b = 0_L$$

$$cx = cx - c_B(B^{-1}Ax - B^{-1}b) = (c - c_B B^{-1}A)x - c_B B^{-1}b$$

$$= (c - c_B B^{-1}A)x - c_B x_B$$

				c_1	$\cdots$	c_k	$\cdots$	c_S			
	Zmienne bazowe	c_B	h_0	h_1	$\dots$	h_k	$\dots$	h_S		$\frac{h_{s0}}{h_{sk}}$	$h_{sk} \geq 0$
	x_{j1}	c_{j1}	h_{10}	h_{11}	$\cdots$	h_{1k}	$\cdots$	h_{1s}			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$\leftarrow$	x_{jl}	c_{jl}	h_{l0}	h_{l1}	$\cdots$	h_{lk}	$\cdots$	h_{ls}			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
	x_{jL}	c_{jL}	h_{L0}	h_{L1}	$\cdots$	h_{Lk}	$\cdots$	h_{Ls}			
				$c_1 - z_1$	$\cdots$	$c_k - z_k$	$\cdots$	$c_s - z_s$			
						$\uparrow$					

$$z_k = \sum_{s \in I_B} c_s h_{sk}$$

$$h'_{ls} := \frac{h_{ls}}{h_{lk}}; \quad h'_{is} = h_{is} - \frac{h_{ik} h_{ls}}{h_{lk}}$$

$$s = 1, 2, \dots, S \qquad i = 0, 1, \dots, S$$

$$s \in I_B \backslash \{l\}$$

The simplex method

1. Generation of initial basis $x_B = B^{-1}b$
2. Checking $c - c_B B^{-1} A \geq 0_S$. If it holds, then x_B is basic feasible solution $x = [x_B \; 0]$
3. Such a k that $c_k - z_k = \min_{1 \leq s \leq S} (c_s - z_s)$ is introduced to the basis, $z_k = \sum_{s \in I_B} c_s h_{sk}$
4. Checking, whether $h_k \leq 0$, if it holds true – solution is unbounded
5. Removing such *l* from the basis, for which:

$$\frac{h_{l0}}{h_{lk}} = \min_{1 \leq s \leq S} \{ \frac{h_{s0}}{h_{sk}}, h_{sk} > 0 \}$$

6. $I_B := I_B \setminus \{l\} \cup \{k\}$

   $I_B = \{j \in \{1, 2, \ldots, S\} \quad x^{(j)}$ belongs to the basis }

7. $c_k - z_k$=0, for k – non basis variable – non unique solution

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			c_1	$\cdots$	c_k	$\cdots$	c_s			
Zmienne bazowe	c_B	h_0	h_1	$\dots$	h_k	$\dots$	h_s		$\frac{h_{s0}}{h_{sk}}$	$h_{sk} \geq 0$
x_{j1}	c_{j1}	h_{10}	h_{11}	$\cdots$	h_{1k}	$\cdots$	h_{1s}			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
← x_{jl}	c_{jl}	h_{l0}	h_{l1}	$\cdots$	h_{lk}	$\cdots$	h_{ls}		$\leftarrow min\, \frac{h_{s0}}{h_{sk}}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
x_{jL}	c_{jL}	h_{L0}	h_{L1}	$\cdots$	h_{Lk}	$\cdots$	h_{Ls}			
			$c_1 - z_1$	$\cdots$	$c_k - z_k$	$\cdots$	$c_s - z_s$			
					↑					

$$z_k = \sum_{s \in I_B} c_s h_{sk}$$

$$h'_{ls} := \frac{h_{ls}}{h_{lk}}; \quad h'_{is} = h_{is} - \frac{h_{ik} h_{ls}}{h_{lk}}$$

$$s = 1, 2, \dots, S \qquad i = 0, 1, \dots, S$$

$$s \in I_B \backslash \{l\}$$

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Example

$$F(x) = -2x^{(1)} - 3x^{(2)} \rightarrow F(x) = -2x^{(1)} - 3x^{(2)} + 0x^{(3)} + 0x^{(4)} + 0x^{(5)}$$

$$\begin{aligned}
2x^{(1)} + 2x^{(2)} - 14 \leq 0 &\rightarrow 2x^{(1)} + 2x^{(2)} + x^{(3)} = 14 \\
x^{(1)} + 2x^{(2)} - 8 \leq 0 &\qquad x^{(1)} + 2x^{(2)} + x^{(4)} = 8 \\
4x^{(1)} - 16 \leq 0 &\qquad 4x^{(1)} + x^{(5)} = 16
\end{aligned}$$

$$F(x) = \begin{bmatrix} -2 & -3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ x^{(4)} \\ x^{(5)} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ x^{(4)} \\ x^{(5)} \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 16 \end{bmatrix}$$

			-2	-3	0	0	0		
Zmienne bazowe	c_B	h_0	h_1	h_2	h_3	h_4	h_5		
x_3	0	14	2	2	1	0	0	$\frac{14}{2}$	
x_4	0	8	1	2	0	1	0	$\frac{8}{2}$	$\leftarrow min \frac{h_{s0}}{h_{sk}}$
x_5	0	16	4	0	0	0	1	-	
		0	-2	-3	0	0	0		

$\uparrow$ (column h_2)

$$I_B = \{3, 4, 5\}$$

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			-2	-3	0	0	0	
Zmienne bazowe	c_B	h_0	h_1	h_2	h_3	h_4	h_5	
x_3	0	6	1	0	1	-1	0	$\frac{6}{1}$
x_2	-3	4	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{4}{\frac{1}{2}}$
x_5	0	16	4	0	0	0	1	$\frac{16}{4}$
			$-\frac{1}{2}$	0	0	$\frac{3}{2}$	0	

$\leftarrow min \frac{h_{s0}}{h_{sk}}$

$\uparrow$

		x_B	**-2**	**-3**	**0**	**0**	**0**	
Zmienne bazowe	c_B	h_0	h_1	h_2	h_3	h_4	h_5	
x_3	0	2	0	0	1	-1	$-\frac{1}{4}$	
x_2	-3	2	0	1	0	$\frac{1}{2}$	$-\frac{1}{8}$	
x_1	-2	4	1	0	0	0	$\frac{1}{4}$	
			0	0	0	$\frac{3}{2}$	$\frac{1}{8}$	

≥ 0

The final solution $[4 \quad 2 \quad 2 \quad 0 \quad 0]$

Finding initial feasible basic solution (additional task)

$$Ax = b$$

$$x \geq 0_S$$

Artificial variables:

$$Ax + Ix_a = b \quad x \geq 0, x_a \geq 0$$

Additional task

$$\min_{x_a} 1^T x_a$$

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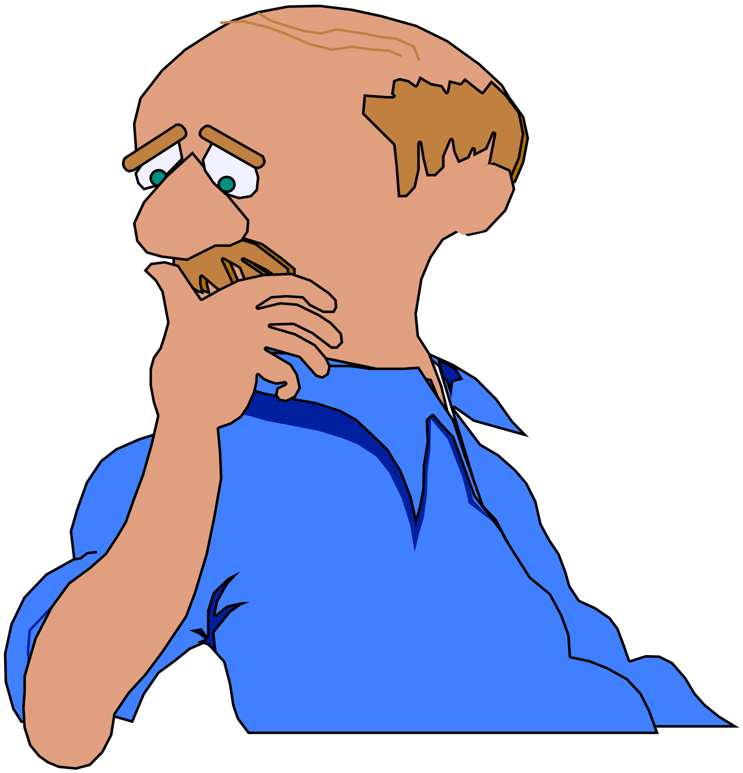
The two phase simplex method

$$F(x) = c^T x + M 1^T x_a$$

Constraints

$$Ax + IX_a = b, x \geq 0, x_A \geq 0$$

Thank you for attention



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Linear programming - standard form

$$F(x) = c^T x$$

A: $\mathcal{D}_X = \{x \in R^S, Ax - b = 0_L, x \geq 0_S\}$

lub

B: $\mathcal{D}_x = \{x \in R^S, Ax - b \leq 0_L, x \geq 0_S\}$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_S \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_L \end{bmatrix}, \quad x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(S)} \end{bmatrix}, \quad A_{SxL} = \begin{bmatrix} a_{11} & \cdots & a_{1S} \\ \vdots & \ddots & \vdots \\ a_{L1} & \cdots & a_{LS} \end{bmatrix}$$

Quadratic programming

$$F(x) = x^T D x + c^T x$$

$$D_x = \{x \in R^s, Ax = b, x \geq 0\}$$

Special Case

$$x^* \to F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : x \geq 0_S, \psi(x) \leq 0_M \right\}$$

$$L(x,\mu) = F(x) + \mu^T \psi(x)$$

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*} \geq 0_S$$

$$x^T \nabla_x L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$x^* \geq 0_S$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} \leq 0_M$$

$$\mu^T \nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$\mu^* \geq 0_M$$

Special Case

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi(x) = 0_L , \psi(x) \le 0_M \right\}$$

$$L(x, \lambda, \mu) = F(x) + \lambda^T \varphi(x) + \mu^T \psi(x)$$

$$\nabla_x L(x, \lambda, \mu)\Big|_{x^*, \lambda^*, \mu^*} = 0_S$$

$$\nabla_\lambda L(x, \lambda, \mu)\Big|_{x^*, \lambda^*, \mu^*} = 0_L$$

$$\mu^T \nabla_\mu L(x, \lambda, \mu)\Big|_{x^*, \lambda^*, \mu^*} = 0$$

$$\nabla_\mu L(x, \lambda, \mu)\Big|_{x^*, \lambda^*, \mu^*} \le 0_M$$

$$\mu^* \ge 0_S$$

Quadratic programming problem

$$F(x) = x^T D x + c^T x$$

$$D_x = \{x \in R^S, Ax = b, x \geq 0\}$$

$$L(x,\lambda) = x^T D x + C^T x + \lambda^T (Ax - b)$$

$$\text{V}= \nabla_x L(x,\lambda) = c + 2Dx - A^T\lambda \geq 0$$

$$x^T \nabla_X L(x,\lambda) \quad = x^T \left(c + 2Dx - A^T\lambda\right) = 0$$

$$\nabla_\lambda L(x,\lambda) \quad = \quad Ax - b = 0$$

$$x \geq 0,\ \text{V} \geq 0$$

$$\begin{cases} \text{V}= \nabla_x L(x,\lambda) \geq 0 \\ x^T \nabla_x L(x,\lambda) = 0 \\ \nabla_\lambda L(x,\lambda) = 0 \\ x \geq 0,\ \text{V} \geq 0 \end{cases}$$

$$\begin{cases} c + 2Dx - A^T\lambda - \text{V} = 0 \\ x^T\ \text{V} = 0 \\ Ax = b \\ x \geq 0,\ \text{V} \geq 0, \end{cases}$$

Quadratic programming problem

$$Ax = b$$

$$2Dx - A^T\lambda - V = -c$$

$$x^T V = 0,$$

$$x \geq 0, V \geq 0$$

$$\begin{bmatrix} A & 0 & 0 \\ 2D & -A^T & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ V \end{bmatrix} = \begin{bmatrix} b \\ -c \end{bmatrix}$$

$$x \geq 0, V \geq 0, x^T V = 0$$

$\lambda -$ $\mp$non defined sigh

Quadratic programming problem

$Bx_B = b$ - base solution

$Ax = b$

$2Dx + A^T\lambda - \text{v} + Eu = -c$ $\quad u$ – artificial variable

$u \geq 0$

D_B - matrix , in which matrix column *D* corresponds to column *A* in *B*

E – diagonal matrix

$$\Delta j = \begin{cases} +1 & -c_j - 2d_{Bj}x_B \geq 0 \\ -1 & -c_j - 2d_{Bj}x_B < 0 \end{cases}$$

$u_j = \left|-c_j - 2d_{Bj}x_B\right|, j = 1, 2, \ldots, S, \lambda = 0, \text{v} = 0$

d_{Bj} - j-th row of matrix D_B

Quadratic programming problem

Let $u_j = \left|-c_j - 2d_{Bj}x_B\right|, j = 1, 2, \ldots, S, \lambda = 0, \mathrm{V} = 0$ we obtain one of solution which can be denoted:

$$\begin{bmatrix} B & 0 \\ 2D_B & E \end{bmatrix}\begin{bmatrix} x_B \\ u \end{bmatrix} = \begin{bmatrix} b \\ -c \end{bmatrix} \qquad \begin{bmatrix} A & 0 & 0 \\ 2D & -A^T & I \end{bmatrix}\begin{bmatrix} x \\ \lambda \\ \mathrm{V} \end{bmatrix} = \begin{bmatrix} b \\ -c \end{bmatrix}$$

Now the artificial variables ought to be removed

The linear programming can be used

$$F(u) = 1^T u$$

With constrains

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ 2D & -A^T & A^T & -I & E \end{bmatrix}\begin{bmatrix} x \\ \lambda' \\ \lambda'' \\ \mathrm{V} \\ u \end{bmatrix} = \begin{bmatrix} b \\ -c \end{bmatrix}$$

$$x \geq 0, \mathrm{V} \geq 0, x^T \mathrm{V} = 0, \lambda' \geq 0, \lambda'' \geq 0,$$

Linear Fractional Programming

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

$$\mathcal{D}_x = \{x \in \mathrm{R}^S : \phi(x) = 0_L,\ \psi(x) \le 0_M, x \ge 0_S\}$$

Goal function: $F(x) = \dfrac{a^T x + b}{c^T x + d}$ $a \in \mathscr{R}^S, b \in \mathscr{R}, c \in \mathscr{R}^S, d \in \mathscr{R}$

Constrains:

$$\varphi_l(x) = p_l^T x - \alpha_l = 0, \quad l = 1,2,\ldots,L \qquad p_l = \begin{bmatrix} p_l^{(1)} \\ p_l^{(2)} \\ \vdots \\ p_l^{(S)} \end{bmatrix}$$

$$\psi_m(x) = q_m^T x - \beta_m \le 0, \quad m = 1,2,\ldots,M \qquad q_l = \begin{bmatrix} q_m^{(1)} \\ q_m^{(2)} \\ \vdots \\ q_m^{(S)} \end{bmatrix}$$

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Linear Fractional Programming

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F(x) = \frac{a^T x + b}{c^T x + d} \qquad a \in \mathcal{R}^S,\ b \in \mathcal{R},\ c \in \mathcal{R}^S, d \in \mathcal{R}$$

$$c^T x + d \neq 0$$

$$\mathcal{D}_x = \{x \in R^S, Ax - e \leq 0_L, x \geq 0_S\}$$

Charnes - Cooper Method

Linear Fractional Programming

1. Let us assume that: $c^T x + d > 0$

Denote by $z = \frac{1}{c^T x + d}$ and $y = zx$

In this case the problem is reduced to:

Minimization of $a^T y + bz$ ⟵ $F(x) = \frac{a^T x + b}{c^T x + d}$ /•z

With constrains: $Ay - ez \leq 0$ ⟵ $Ax - e \leq 0_L / \bullet z$

$$c^T y + dz = 1$$

$$y \geq 0$$

$$z \geq 0$$

The problem is reduced to the linear programing task

Linear Fractional Programming

2. Let us assume that: $c^T x + d < 0$

Denote by $-z = \frac{1}{c^T x + d}$ oraz $y = zx$

In this case the problem is reduced to:

Minimization of $-a^T y - bz$ $\longleftarrow$ $F(x) = \frac{a^T x + b}{c^T x + d}$ /•z

Wth constrains: $Ay - ez \leq 0$ $\longleftarrow$ $Ax - e \leq 0_L / \bullet z$

$$-c^T x - dz = 1$$

$$y \geq 0$$

$$z \geq 0$$

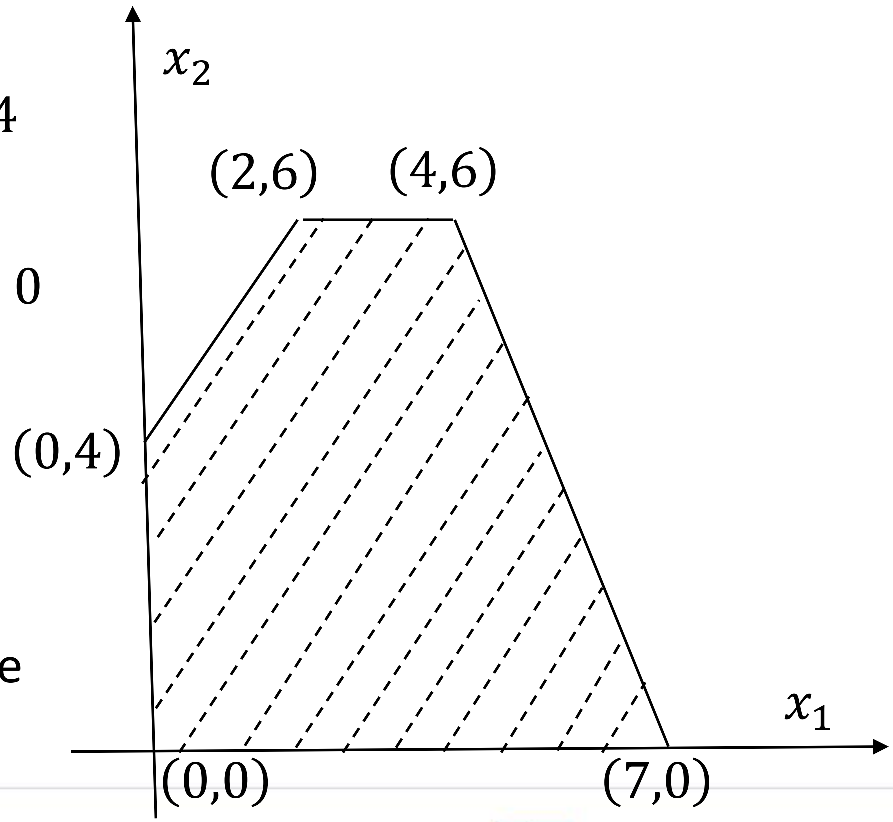
The problem is reduced to the linear programing task

Finally $x = \frac{y}{z}$

Linear Fractional Programming - Example

Goal function $$F(x_1, x_2) = \frac{-2x_1 + x_2 + 2}{x_1 + 3x_2 + 4}$$

Constraints
$$-x_1 + x_2 \leq 4$$
$$2x_1 + x_2 \leq 14$$
$$x_2 \leq 6$$
$$x_1 \geq 0 \; , x_2 \geq 0$$



For the point from feasible set

$$x_1 + 3x_2 + 4 > 0$$

For example for point $(0,0)$ we have

$$0 + 3 \cdot 0 + 4 = 4 > 0$$

Linear Fractional Programming - Example

Let: $z = \frac{1}{x_1 + 3x_2 + 4}, \quad y_1 = zx_1, \quad y_2 = zx_2$

$$F(x_1, x_2) = \frac{-2x_1 + x_2 + 2}{x_1 + 3x_2 + 4} \longrightarrow \bar{F}(y_1, y_2, z) = -2y_1 + y_2 + 2z$$

$$-x_1 + x_2 \leq 4 \;/\cdot z \longrightarrow -y_1 + y_2 - 4z \leq 0$$

$$2x_1 + x_2 \leq 14/\cdot z \longrightarrow 2y_1 + y_2 - 14z \leq 0$$

$$x_2 \leq 6 \;/\cdot z \longrightarrow y_2 - 6z \leq 0$$

$$z = \frac{1}{x_1 + 3x_2 + 4} \;/\cdot\; (x_1 + 3x_2 + 4) \longrightarrow y_1 + 3y_2 + 4z = 1$$

$$x_1 \geq 0 \quad x_2 \geq 0 \longrightarrow y_1 \geq 0 \quad y_2 \geq 0 \quad z \geq 0$$

From linear programming solution we have $\longrightarrow y_1 = 7/11, \; y_2 = 0, \; z = 1/11$

After substitution $\longrightarrow x_1 = \frac{y_1}{z} = 7, \; x_2 = \frac{y_2}{z} = 0$

Thank you for attention